

Thermodynamics, geometrothermodynamics and critical behavior of (2+1)-dimensional black holes

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Abstract

In this paper, we study the properties of the (2+1)-dimensional black holes from the viewpoint of geometrothermodynamics. We show that the Legendre invariant metric of the (2+1)-dimensional black holes can produce correctly the behavior of the thermodynamic interaction and phase transition structure of the corresponding black hole configurations. We find that they are both curved and the curvature scalar gives the information about the phase transition point.

Keywords: Black hole Legendre invariance Curvature scalar Phase transition

1. Introduction

The black hole thermodynamics has been one of the focuses in theoretical physics during the past thirty years [1-9]. The results showed that a black hole is a thermodynamics system, it has Hawking temperature proportional to its surface gravity on the horizon, and they satisfy the four laws of black hole thermodynamics. However, in geometry framework, black hole thermodynamics has been investigated from the critical points of moduli space by using the Weinhold metric and Ruppeiner metric [10]. As is well known, an interesting inner product on the equilibrium thermodynamic space of state in the energy representation was proposed by Weinhold as the Hessian matrix of the internal energy U with respect to the extensive thermodynamic variables N^a , namely $g_{ij}^W = \partial_i \partial_j M(U, N^a)$ [11]. However, there was no physical interpretation associated with this metric structure. As a modification, Ruppeiner introduced Riemannian metric into thermodynamic system once more

and defended it as the second derivative of entropy S (here, entropy is a function of internal energy U and its extensive variables N^a) $g_{ij}^R = -\partial_i \partial_j S(U, N^a)$ [12]. In the next, it was applied to all kinds of thermodynamics modes. For example, Cai and Cho [13] gave a brief review on the geometrical method on the thermodynamics, and applied this approach to the BTZ black hole. Aman et. al. [14], showed curvature scalars and phase transitions of the BTZ and the Reissner-Nordstrom. In addition, Ruppeiner has given a systematic discussion on how to make the correct choice of a metric, and has also demonstrated several limiting results matching extreme Kerr-Newman black hole thermodynamics to the 2- dimensional Fermi gas. This shows that the connection to a 2D model is consistent with the membrane paradigm of black holes [15-16]. Using the Ruppeiners thermodynamics geometry theory, one have shown that Ruppeiner geometry can be carried out in various thermodynamic systems [17-26]. Such as the ideal gas, the van der Waals gas and so on. It was shown that the scalar curvature is zero and the Ruppeiner metric is flat for the van der Waals gas. The curvature is nonzero and diverges only after the phase transition takes place. The key of the above problems is the thermodynamic potential, which is generally believed to be the internal energy rather than the mass. Above researches have shown that Weinholds and Ruppeiners thermodynamic metrics are not invariant under the Legendre transformations.

Recently, Quevedo et al. [27] present a new formalism of geometrothermodynamics (GTD) as a geometric approach that incorporates Legendre invariance in a natural way, and allows us to derive Legendre invariant metrics in the space of equilibrium states. Considering the Legendre invariant, they present a unified geometry where the metric structure can give a well description of various types of black hole thermodynamics [28-31]. The aim of the application of different thermodynamic geometries is to describe phase transitions in terms of curvature singularities. For a thermodynamic system, it is quite interesting to investigate the corresponding relationship between the curvature of Weinhold metric, Ruppeiner metric, the Legendre invariant metric and the phase transitions. In fact, above viewpoint has been applied to various black holes [19, 26]. Of course, it is still widely believed that the thermodynamic geometry of a black hole is still a most fascinating and unresolved subject today. The main purpose of the present work is to show that the Legendre invariant metric can be used to reproduce correctly the thermodynamics of the (2+1)-dimensional black holes. This has been analyzed previously by using a different approach where Legendre invariance is

not taken into account [32].

The organization of the Letter is outlined as follows. In Sec. 2, we present a (2+1)-dimensional black hole with a coulomb-like field. In Sec.3, show geometrothermodynamics of the (2+1)-dimensional black hole with a coulomb-like field. Sec. 4 ends up with some discussions and conclusions. Throughout the Letter, the units $c = k_B = \hbar = 1$ are used.

2. The (2+1)-dimensional black hole with a coulomb-like field

The action describing the (2+1)-dimensional Einstein theory coupled with nonlinear electrodynamics is given by [33]

$$S = \int \sqrt{g} \left(\frac{1}{16\pi} \right) (R - 2\Lambda) + L(F) d^3x, \quad (1)$$

with arbitrary, at this stage, the electromagnetic Lagrangian $L(F)$. We are using units in which $c = G = 1$. Since there is a T ambiguity in the definition of the gravitational constant there is not Newtonian gravitational limit in 2+1 dimensions one can maintain the factor $\frac{1}{16\pi}$ in the action to keep the parallelism with 3+1-gravity. The variation with respect to the metric gives the Einstein equations

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}, \quad (2)$$

$$T_{ab} = g_{ab} L(F) - F_{ac} F_b^c L_{,F}, \quad (3)$$

$$\nabla_a (F^{ab} L_{,F}) = 0, \quad (4)$$

where stands $L_{,F}$ for the derivative with respect to $F = (F_{ab} F^{ab})/4$. The nonlinear field is chosen such that the energy momentum tensor (3) has a vanishing trace. The trace of the tensor gives

$$T = T_{ab} g^{ab} = 3L(F) - 4F L_{,F}. \quad (5)$$

In order to have a vanishing trace, the electromagnetic Lagrangian is obtained as

$$L = C |F|^{3/4}, \quad (6)$$

where C is an integration constant. One can rewrite this Lagrangian as

$$L = C \left| \frac{1}{2} (B^2 - E^2) \right| = 1, \quad (7)$$

when referred to orthonormal local Lorentzian basis. With reference to the paper [34], the complete solution to the above action is given by the metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\theta^2, \quad (8)$$

where the metric function is $f(r)$ given by

$$f(r) = -M + \frac{r^2}{l^2} + \frac{4Q^2}{3r}. \quad (9)$$

Here M is the mass, $l^2 = \Lambda^{-1}$ the case $\Lambda > 0$ ($\Lambda < 0$), corresponds to an asymptotically de-Sitter (anti de-Sitter) space-time, Q is the electric charge. From Eq. (9), the event horizon is located at $f(r_h) = 0$ and the radius r_h satisfies

$$M = \frac{r_h^2}{l^2} + \frac{4Q^2}{3r_h}. \quad (10)$$

For the extremal black hole, there exist two event horizons, the inner event horizon and the outer event horizon. Here, we have denoted r_h as the radius of outer event horizon. From the energy conservation law of the black hole

$$dM = TdS + \phi dQ, \quad (11)$$

Using the relation between entropy and the radius of the event horizon, we can obtain

$$S = 4\pi r_h. \quad (12)$$

The thermodynamic temperature and electric potential can be expressed

$$T = \left(\frac{\partial M}{\partial S}\right)_Q = \frac{S}{8l^2\pi^2} - \frac{16\pi Q^2}{3S^2} = \frac{1}{2\pi} \left(\frac{r_h}{l^2} - \frac{2Q^2}{3r_h^2}\right). \quad (13)$$

and

$$\phi = \left(\frac{\partial M}{\partial Q}\right)_S = \frac{32\pi Q}{3S}. \quad (14)$$

3. Geometrothermodynamics of the (2+1)-dimensional black hole with a coulomb-like field

Now, we turn to the recent geometric formulation of extended thermodynamic behavior of the (2+1)-dimensional black hole with a coulomb-like field.

The formulation of GTD of black hole is based on the theory of contact geometry as a framework for thermodynamics [27]. Consider the $(2n+1)$ -dimensional thermodynamic phase space \mathfrak{J} with the coordinates $Z^A = \{\Phi, E^a, I^a\}$ where $A = 0, \dots, 2n$ and $a = 1, \dots, n$. In ordinary thermodynamics, Φ corresponds to the thermodynamic potential, and E^a, I^a are the extensive and intensive variables, respectively. The fundamental differential form Θ can then be written in a canonical manner as $\Theta = d\Phi - \delta_{ab} I^a dE^b$, where δ_{ab} is the Euclidean metric. Considering a non-degenerate metric $G = G(Z^A)$, and the Gibbs 1-form, with $\delta_{ab} = \text{diag}\{1, \dots, 1\}$, we obtain a set $(\mathfrak{J}, \Theta, G)$ which defines a contact Riemannian manifold if the condition $\Theta \wedge (d\Theta)^n \neq 0$ is satisfied. This arbitrariness is restricted by the condition that G must be invariant with respect to Legendre transformations. This is a necessary condition for our description of thermodynamic systems to be independent of the thermodynamic potential. This implies that T must be a curved manifold [27] because the special case of a metric with vanishing curvature turns out to be non-Legendre invariant. The Gibbs 1-form Θ is also invariant with respect to Legendre transformations. Legendre invariance guarantees that the geometric properties of G do not depend on the thermodynamic potential.

The thermodynamic phase space \mathfrak{J} with a coulomb-like field can be defined as a 5-dimensional space with coordinates $Z^A = \{M, S, T, Q\}$, $A = 0, \dots, 4$. The Eq. (10) represents the fundamental relationship $M(S, Q)$ from which all the thermodynamic information can be obtained. Therefore, we would like to consider a 5-dimensional phase space \mathfrak{J} with coordinates (M, S, T, Q, Φ) , a contact 1-form

$$\Theta = dM - TdS - \phi dQ, \quad (15)$$

and an invariant metric

$$G = (dM - TdS - \phi dQ)^2 + (TS + \phi Q)(-dTdS + d\phi dQ). \quad (16)$$

The triplet $(\mathfrak{J}, \Theta, G)$ defines a contact Riemannian manifold that plays an auxiliary role in GTD. We should properly handle the invariance with respect to Legendre transformations. In fact, for the charged black hole, a Legendre transformation involves in general all the thermodynamic variables M, S, Q, T and ϕ . So they must be independent from each other as they are in the phase space. We introduce also the geometric structure of the space of equilibrium states ε in the following manner: ε is a 2-dimensional submanifold of \mathfrak{J} that is defined by the smooth embedding map $\varphi : \varepsilon \mapsto \mathfrak{J}$,

which satisfies the condition that the projection of the contact form Θ on ε vanishes, namely $\varphi^*(\Theta) = 0$, where φ^* is the pullback of φ . G induces a Legendre invariant metric g on ε by means of ε . In principle, any 2-dimensional subset of the set of coordinates of \mathfrak{J} can be used in coordinative ε . For the sake of simplicity, we will use the set of extensive variables s and Q which in ordinary thermodynamics corresponds to the energy representation. Then, the embedding map for this specific choice is

$$\varphi : \{S, Q\} \mapsto \{M(S, Q), S, Q, \frac{\partial M}{\partial S}, \frac{\partial M}{\partial Q}\}. \quad (17)$$

The condition $\varphi^*(\Theta) = 0$ is equivalent to Eq. (11) (the first law of thermodynamics), Eq. (13), Eq. (14) (the conditions of thermodynamic equilibrium). Then the induced metric is obtained

$$g = (S \frac{\partial M}{\partial S} + Q \frac{\partial M}{\partial Q}) (-\frac{\partial^2 M}{\partial S^2} dS^2 + \frac{\partial^2 M}{\partial Q^2} dQ^2). \quad (18)$$

This metric determines all the geometric properties of the equilibrium space ε . We see that in order to obtain the explicit form of the metric it is necessary to specify the thermodynamic potential M as a function of S and Q . In ordinary thermodynamics this function is usually referred to as the fundamental equation from which all the equations of state can be derived.

Substituting Eq. (12) into Eq. (10), the mass can be obtained as the function of the entropy S and the charge Q in the form

$$M(S, Q) = \frac{S^2}{16\pi^2 l^2} + \frac{16\pi Q^2}{3S}. \quad (19)$$

It has been established that the physical parameters of the (2+1)-dimensional black hole with nonlinear electrodynamics satisfy the first law of black hole thermodynamics.

Substituting Eq. (19) into Eq. (18), we can obtain the Legendre metric components of the (2+1)-dimensional black hole with a coulomb-like field as

$$g_{SS} = -\frac{512\pi^3 Q^4}{9S^4} - \frac{2Q^2}{\pi l^2} - \frac{S^2}{64\pi^4 l^4}, \quad (20)$$

$$g_{QQ} = \frac{512\pi^2 Q^2}{9S^2} + \frac{4S}{3\pi l^2}. \quad (21)$$

After some calculations, we obtain the Legendre invariant scalar curvature

$$\mathfrak{R}_L = \frac{864\pi^4 S^5 l^4 (425984\pi^6 Q^4 l^4 + 1152\pi^3 Q^2 S^3 l^2 - 81S^6)}{(3S^3 + 128\pi^3 Q^2 l^2)^3 (3S^3 + 256\pi^3 Q^2 l^2)^2}. \quad (22)$$

The curved nature of the Legendre metric suggests that the thermodynamics of the present black hole has statistical mechanics analogue.

Now, for a given charge, the heat capacity has the expression

$$C_Q = T \left(\frac{\partial S}{\partial T} \right)_Q = \frac{S(3S^2 - 128\pi^3 Q^2 l^2)}{3S^2 + 256\pi^3 Q^2 l^2}. \quad (22)$$

Obviously, the heat capacities have the zero-points at $3S^2 = 128\pi^3 Q^2 l^2$. Moreover, C_Q changes sign and the scalar curvature diverge at $3S^2 = -256\pi^3 Q^2 l^2$. Therefore, there will be a phase transition at $3S^2 = -256\pi^3 Q^2 l^2$.

4. Conclusion and Discussion

In this work we reproduced the thermodynamics properties such as temperature and entropy of the (2+1)-dimensional black holes. We also studied the Legendre invariant metric of the (2+1)-dimensional black holes. The results show that GTD delivers a particular thermodynamic metric for the (2+1)-dimensional black holes. Then we could corroborate that the thermodynamic curvature is nonzero and its singularities reproduce the phase transition structure which follows from the divergencies of the heat capacity.

In addition, the thermodynamic metric proposed in this work has been applied to the case of black hole configurations in three dimensions. It has been shown that this thermodynamic metric correctly describes the thermodynamic behavior of the corresponding black hole configurations. One additional advantage of this thermodynamic metric is its invariance with respect to total Legendre transformations. This means that the results are independent on the thermodynamic potential used to generate the thermodynamic metric. In all the remaining cases, the singularities of the thermodynamic curvature correspond to points where the heat capacity diverges and phase transitions take place. We interpret this result as an additional indication that the thermodynamic curvature, as defined in GTD, can be used as measure of thermodynamic interaction. In fact, it has been shown that in the case of more realistic thermodynamic systems [30], the ideal gas is also characterized by a vanishing thermodynamic curvature, whereas the van der Waals

gas generates a nonvanishing curvature whose singularities reproduces the corresponding phase transition structure.

Furthermore, we expect that this unified geometry description may give more information about a thermodynamic system. We conclude that GTD is, in general, duality invariant. Therefore, our results support Quevedo's viewpoint.

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